**EJERCICIO (33:52)**

**Calcular los vectores y valores propios de la matriz**

$$\boldsymbol{h_{fotón}} = \begin{pmatrix} 0 & -i\cos\theta & i\sin\theta\sin\phi \\ i\cos\theta & 0 & -i\sin\theta\cos\phi \\ -i\sin\theta\sin\phi & i\sin\theta\cos\phi & 0 \end{pmatrix}$$

Calculamos los valores propios haciendo:

$$\det\left(h_{fotón} - \lambda\mathbb{I}\right) = 0$$

$$\det\begin{pmatrix} -\lambda & -i\cos\theta & i\sin\theta\sin\phi \\ i\cos\theta & -\lambda & -i\sin\theta\cos\phi \\ -i\sin\theta\sin\phi & i\sin\theta\cos\phi & -\lambda \end{pmatrix} = 0$$

$$-\lambda(\lambda^2 + i\sin\theta\cos\phi\, i\sin\theta\cos\phi) + i\cos\theta\,(-\lambda\, i\cos\theta - i\sin\theta\cos\phi\, i\sin\theta\sin\phi) + i\sin\theta\sin\phi\,(i\cos\theta\, i\sin\theta\cos\phi - \lambda\, i\sin\theta\sin\phi) = 0$$

$$-\lambda^3 + \lambda\,(\sin\theta)^2(\cos\phi)^2 + \lambda(\cos\theta)^2 + i\,(\sin\theta)^2\cos\phi\sin\phi\cos\theta - i\,(\sin\theta)^2\cos\phi\sin\phi\cos\theta + \lambda\,(\sin\theta)^2\,(\sin\phi)^2 = 0$$

$$-\lambda^3 + \lambda\,(\sin\theta)^2(\cos\phi)^2 + \lambda(\cos\theta)^2 + \lambda\,(\sin\theta)^2\,(\sin\phi)^2 = 0$$

$$\lambda(\lambda^2 - (\sin\theta)^2((\cos\phi)^2 + (\sin\phi)^2) - (\cos\theta)^2) = 0$$

$$\lambda(\lambda^2 - (\sin\theta)^2 - (\cos\theta)^2) = 0$$

$$\lambda(\lambda^2 - 1) = 0$$

Los autovalores resultan:

$$\lambda_{+1} = 1$$

$$\lambda_0 = 0$$

$$\lambda_{-1} = -1$$

Calculamos ahora los autovectores

Para $\lambda_{+1} = 1$

$$\widehat{e_{+1}} = \begin{pmatrix} {e_{+1}}^1 \\ {e_{+1}}^2 \\ {e_{+1}}^3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -i\cos\theta & i\sin\theta\sin\phi \\ i\cos\theta & -1 & -i\sin\theta\cos\phi \\ -i\sin\theta\sin\phi & i\sin\theta\cos\phi & -1 \end{pmatrix}\begin{pmatrix} {e_{+1}}^1 \\ {e_{+1}}^2 \\ {e_{+1}}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -{e_{+1}}^1 - i\cos\theta\,{e_{+1}}^2 + i\sin\theta\sin\phi\,{e_{+1}}^3 = 0 \\ i\cos\theta\,{e_{+1}}^1 - {e_{+1}}^2 - i\sin\theta\cos\phi\,{e_{+1}}^3 = 0 \\ -i\sin\theta\sin\phi\,{e_{+1}}^1 + i\sin\theta\cos\phi\,{e_{+1}}^2 - {e_{+1}}^3 = 0 \end{cases}$$

$${e_{+1}}^1 = -i\cos\theta\,{e_{+1}}^2 + i\sin\theta\sin\phi\,{e_{+1}}^3$$

Reemplazando:

$$i\cos\theta\,(-i\cos\theta\,{e_{+1}}^2 + i\sin\theta\sin\phi\,{e_{+1}}^3) - {e_{+1}}^2 - i\sin\theta\cos\phi\,{e_{+1}}^3 = 0$$

$$(\cos\theta)^2{e_{+1}}^2 + i\sin\theta\sin\phi\, i\cos\theta\,{e_{+1}}^3 - {e_{+1}}^2 - i\sin\theta\cos\phi\,{e_{+1}}^3 = 0$$

$$e_{+1}{}^2 = \frac{-i\sin\theta\sin\phi\, i\cos\theta + i\sin\theta\cos\phi}{(\cos\theta)^2 - 1} e_{+1}{}^3 = \frac{\sin\theta}{-(\sin\theta)^2}(-i\sin\phi\, i\cos\theta + i\cos\phi)e_{+1}{}^3$$

$$e_{+1}{}^2 = -(\sin\phi\cos\theta + i\cos\phi)\frac{e_{+1}{}^3}{\sin\theta}$$

Reemplazando:

$$e_{+1}{}^1 = -i\cos\theta\left(-(\sin\phi\cos\theta + i\cos\phi)\frac{e_{+1}{}^3}{\sin\theta}\right) + i\sin\theta\sin\phi\, e_{+1}{}^3$$

$$e_{+1}{}^1 = (i\cos\theta\,(\sin\phi\cos\theta + i\cos\phi) + i\,(\sin\theta)^2\sin\phi)\frac{e_{+1}{}^3}{\sin\theta}$$

$$e_{+1}{}^1 = (i(\cos\theta)^2\sin\phi + \cos\theta\cos\phi + i\,(\sin\theta)^2\sin\phi)\frac{e_{+1}{}^3}{\sin\theta}$$

$$e_{+1}{}^1 = (i\sin\phi - \cos\theta\cos\phi)\frac{e_{+1}{}^3}{\sin\theta}$$

Para simplificar adoptamos $e_{+1}{}^3 = -\sin\theta$

$$\begin{cases} e_{+1}{}^1 = \cos\theta\cos\phi - i\sin\phi \\ e_{+1}{}^2 = \sin\phi\cos\theta + i\cos\phi \\ e_{+1}{}^3 = -\sin\theta \end{cases}$$

Normalizamos haciendo

$$e_{+1}\cdot e_{+1} = e_{+1}{}^{1^*}\times e_{+1}{}^1 + e_{+1}{}^{2^*}\times e_{+1}{}^2 + e_{+1}{}^{2^*}\times e_{+1}{}^2$$

$$e_{+1}\cdot e_{+1} = (\cos\theta)^2(\cos\phi)^2 + (\sin\phi)^2 + (\sin\phi)^2(\cos\theta)^2 + (\cos\phi)^2 + (\sin\theta)^2$$

$$e_{+1}\cdot e_{+1} = (\cos\theta)^2 + 1 + (\sin\theta)^2 = 2$$

El vector propio para:

$$\boxed{\boldsymbol{\lambda_{+1} = 1 \longrightarrow \widehat{e_{+1}} = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta\cos\phi - i\sin\phi \\ \sin\phi\cos\theta + i\cos\phi \\ -\sin\theta\end{pmatrix}}}$$

Para $\lambda_0 = 0$

$$\widehat{e_0} = \begin{pmatrix} e_0{}^1 \\ e_0{}^2 \\ e_0{}^3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -i\cos\theta & i\sin\theta\sin\phi \\ i\cos\theta & 0 & -i\sin\theta\cos\phi \\ -i\sin\theta\sin\phi & i\sin\theta\cos\phi & 0 \end{pmatrix}\begin{pmatrix} e_0{}^1 \\ e_0{}^2 \\ e_0{}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 - i\cos\theta\, e_0{}^2 + i\sin\theta\sin\phi\, e_0{}^3 = 0 \\ i\cos\theta\, e_0{}^1 - 0 - i\sin\theta\cos\phi\, e_0{}^3 = 0 \\ -i\sin\theta\sin\phi\, e_0{}^1 + i\sin\theta\cos\phi\, e_0{}^2 - 0 = 0 \end{cases}$$

$$e_0{}^2 = \sin\theta\sin\phi\frac{e_0{}^3}{\cos\theta}$$

$$e_0{}^1 = \sin\theta\cos\phi\frac{e_0{}^3}{\cos\theta}$$

Para simplificar adoptamos ${e_0}^3 = \cos\theta$

$$\begin{cases} {e_0}^1 = \sin\theta\sin\phi \\ {e_0}^2 = \sin\theta\cos\phi \\ {e_0}^3 = \cos\theta \end{cases}$$

Normalizamos haciendo

$$e_0 \cdot e_0 = {e_0}^{1^*} \times {e_0}^1 + {e_0}^{2^*} \times {e_0}^2 + {e_0}^{2^*} \times {e_0}^2$$

$$e_0 \cdot e_0 = (\sin\theta)^2(\sin\phi)^2 + (\sin\theta)^2(\cos\phi)^2 + (\cos\theta)^2$$

$$e_0 \cdot e_0 = (\sin\theta)^2 + (\cos\theta)^2 = 1$$

El vector propio para:

$$\boxed{\boldsymbol{\lambda_0 = 1 \longrightarrow \widehat{e_0} = \begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix}}}$$

Para $\lambda_{-1} = -1$

$$\widehat{e_{-1}} = \begin{pmatrix} {e_{-1}}^1 \\ {e_{-1}}^2 \\ {e_{-1}}^3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -i\cos\theta & i\sin\theta\sin\phi \\ i\cos\theta & 1 & -i\sin\theta\cos\phi \\ -i\sin\theta\sin\phi & i\sin\theta\cos\phi & 1 \end{pmatrix} \begin{pmatrix} {e_{-1}}^1 \\ {e_{-1}}^2 \\ {e_{-1}}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} {e_{-1}}^1 - i\cos\theta\, {e_{-1}}^2 + i\sin\theta\sin\phi\, {e_{-1}}^3 = 0 \\ i\cos\theta\, {e_{-1}}^1 + {e_{-1}}^2 - i\sin\theta\cos\phi\, {e_{-1}}^3 = 0 \\ -i\sin\theta\sin\phi\, {e_{-1}}^1 + i\sin\theta\cos\phi\, {e_{-1}}^2 + {e_{-1}}^3 = 0 \end{cases}$$

$${e_{-1}}^1 = i\cos\theta\, {e_{-1}}^2 - i\sin\theta\sin\phi\, {e_{-1}}^3$$

Reemplazando:

$$i\cos\theta\,(i\cos\theta\, {e_{-1}}^2 - i\sin\theta\sin\phi\, {e_{-1}}^3) + {e_{-1}}^2 - i\sin\theta\cos\phi\, {e_{-1}}^3 = 0$$

$$-(\cos\theta)^2 {e_{-1}}^2 - i\sin\theta\sin\phi\, i\cos\theta\, {e_{-1}}^3 - {e_{-1}}^2 - i\sin\theta\cos\phi\, {e_{-1}}^3 = 0$$

$${e_{-1}}^2 = \frac{+i\sin\theta\sin\phi\, i\cos\theta + i\sin\theta\cos\phi}{-(\cos\theta)^2 + 1}\, {e_{-1}}^3 = \frac{\sin\theta}{(\sin\theta)^2}(i\sin\phi\, i\cos\theta + i\cos\phi){e_{-1}}^3$$

$${e_{-1}}^2 = (-\sin\phi\cos\theta + i\cos\phi)\frac{{e_{-1}}^3}{\sin\theta}$$

Reemplazando:

$${e_{-1}}^1 = -i\cos\theta\left((-\sin\phi\cos\theta + i\cos\phi)\frac{{e_{-1}}^3}{\sin\theta}\right) - i\sin\theta\sin\phi\, {e_{-1}}^3$$

$${e_{-1}}^1 = (i\cos\theta\,(-\sin\phi\cos\theta + i\cos\phi) - i\,(\sin\theta)^2\sin\phi)\frac{{e_{-1}}^3}{\sin\theta}$$

$${e_{-1}}^1 = (-i(\cos\theta)^2\sin\phi - \cos\theta\cos\phi - i\,(\sin\theta)^2\sin\phi)\frac{{e_{-1}}^3}{\sin\theta}$$

$$e_{\text{-}1}{}^{1} = (-i\sin\phi - \cos\theta\cos\phi)\frac{e_{\text{-}1}{}^{3}}{\sin\theta}$$

Para simplificar adoptamos $e_{\text{-}1}{}^{3} = -\sin\theta$

$$\begin{cases} e_{\text{-}1}{}^{1} = \cos\theta\cos\phi + i\sin\phi \\ e_{\text{-}1}{}^{2} = \sin\phi\cos\theta - i\cos\phi \\ e_{\text{-}1}{}^{3} = -\sin\theta \end{cases}$$

Normalizamos haciendo

$$e_{\text{-}1}\cdot e_{\text{-}1} = e_{\text{-}1}{}^{1^*}\times e_{\text{-}1}{}^{1} + e_{\text{-}1}{}^{2^*}\times e_{\text{-}1}{}^{2} + e_{\text{-}1}{}^{2^*}\times e_{\text{-}1}{}^{2}$$

$$e_{\text{-}1}\cdot e_{\text{-}1} = (\cos\theta)^2(\cos\phi)^2 + (\sin\phi)^2 + (\sin\phi)^2(\cos\theta)^2 + (\cos\phi)^2 + (\sin\theta)^2$$

$$e_{\text{-}1}\cdot e_{\text{-}1} = (\cos\theta)^2 + 1 + (\sin\theta)^2 = 2$$

El vector propio para:

$$\boxed{\boldsymbol{\lambda_{-1} = 1 \longrightarrow \widehat{e_{-1}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta\cos\phi + i\sin\phi \\ \sin\phi\cos\theta - i\cos\phi \\ -\sin\theta \end{pmatrix}}}$$